Series

In this chapter, we define series of complex numbers. Whenever we encounter series of real numbers, we will freely use results that are proved in calculus.

Our main interest is the representation of functions as infinite series. Among other thing, we will prove:

(1) A function f that is analytic on a disk $D_R(z_0)$ has a convergent power series representation on that disk:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k .$$

Conversely, every power series $\sum_{k=0}^{\infty} a_n (z-z_0)^k$ is an analytic function on its domain of convergence.

(2) A function that is analytic on an annulus $R_1 < |z-z_0| < R_2$ has a convergent series representation on that annulus:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} b_k \frac{1}{(z-z_0)^k}$$

with coefficients

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{k+1}} dz \quad \text{and} \quad b_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-k+1}} dz$$

where C is any pos. oriented simple closed contour in the annulus and surrounding z_0.

In fact, (2) provides a method for computing integrals over contours that surround a singular point! Suppose that f has a singularity at z_0, but is analytic everywhere else in a deleted neighborhood $D_R(z_0)\setminus\{z_0\}$ of z_0. Then f is analytic on the annulus $0<|z-z_0|<R$. If C is any simple closed positively oriented contour around z_0 and lying inside the annulus, then according to (2),

$$2\pi i \, b_1 = \int_C f(z)\, dz.$$

In other words, we can compute the contour integral of f about a singularity just by computing the coefficient b_1 in the series! This is the beginning of the Theory of Residues.

Sequences

Definition (Sequences) A sequence of complex numbers is a complex-valued function z whose domain is the set of positive integers $\mathbb{N}$. We write $z_n = z(n)$ for the value of z at $n \in \mathbb{N}$. We think of the values as occurring in a certain order:

$$z_1, z_2, z_3, \ldots, z_n, \ldots$$

//

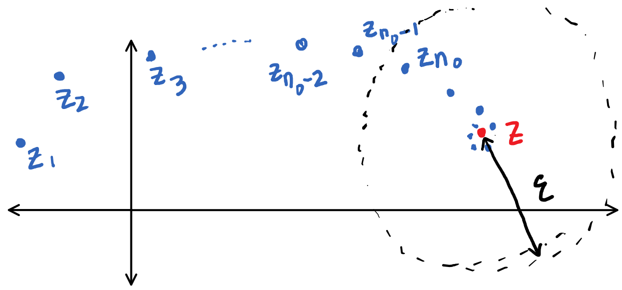
Definition (Limit of a Sequence) A sequence z_n has a limit $z \in \mathbb{C}$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \text{implies} \quad |z_n - z| < \varepsilon.$$

A sequence that has a limit is convergent and we write

$$\lim_{n\to\infty} z_n = z.$$

A sequence with no limit is divergent.



$\boxed{\text{Proposition}}$

(1) The limit of a convergent sequence is unique.

(2) If $z_n = x_n + i y_n$ is a sequence, then

$$\lim_{n\to\infty} x_n + iy_n = x + iy \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Proof.

(1) Assume $\lim_{n\to\infty} z_n = z_1$ and $\lim_{n\to\infty} z_n = z_2$. Let $\varepsilon > 0$. Choose $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \implies |z_n - z_1| < \frac{\varepsilon}{2}$$

Then $n \geq n_2 \implies |z_n - z_2| < \frac{\varepsilon}{2}$, $n = \max\{n_1, n_2\}$.

$$|z_1 - z_2| \leq |z_n - z_1| + |z_n - z_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(2) ($\Rightarrow$) Assume $\lim_{n\to\infty} x_n + iy_n = x + iy$. Let $\varepsilon > 0$. Choose

$n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n + iy_n - (x+iy)| < \varepsilon$. $|x_n - x| \leq |x_n - x + i(y_n - y)|$

But then $|y_n - y| \leq |x_n - x + i(y_n - y)| < \varepsilon$.

($\Leftarrow$) Similar to the argument in (1).

■

$\boxed{\text{Example}}$ We show that

$$\lim_{n\to\infty} -1 + i\,\frac{(-1)^n}{n^2} = -1.$$

By the theorem

$$\frac{}{n^2} = \lim_{n\to\infty} -1 + i \lim_{n\to\infty} \frac{(-1)^n}{n^2}$$

$$\lim_{n\to\infty} -1 + i\,(-1)^n = -1 + 0 = -1.$$

By definition, let $\varepsilon > 0$. Choose $n_0 > \frac{1}{\sqrt{\varepsilon}}$. Then for an $n \geq n_0$,

$$\left| -1 + i\,\frac{(-1)^n}{n^2} - (-1) \right| = \left| i\,\frac{(-1)^n}{n^2} \right| = \frac{1}{n^2} < \varepsilon.$$

//

$\boxed{\text{Definition (Series)}}$ A series of complex numbers is a symbol

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

associated to a sequence z_n of complex numbers. A series has an associated sequence of partial sums

$$S_N = \sum_{n=1}^{N} z_n = \underbrace{z_1 + z_2 + \cdots + z_N}_{\text{sum the first } N \text{ terms}}.$$

A series is convergent if S_N is convergent. In this case, we write

$$\sum_{n=1}^{\infty} z_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} z_n.$$

The limit $\lim_{N\to\infty} S_N$ is called the sum of the series. A series that does not converge is **divergent**. //

Proposition Suppose that $z_n = x_n + i y_n$ is a sequence. Then

$$\sum_{n=1}^{\infty} z_n = X + iY \iff \sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

Proof. This is just the proposition for sequences applied to the partial sums. $\square$

According to the proposition, we can write

$$\sum_{n=1}^{\infty} x_n + i y_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

provided that the series on the left converges <u>or</u> the two on the right converge.

Several results from calculus have counterparts in complex analysis:

Proposition (Test for Divergence) If $\sum_{n=1}^{\infty} z_n$ converges, then

$$\lim_{n\to\infty} z_n = 0.$$

Proof. Write $z_n = x_n + i y_n$. Then by the proposition, the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge. But these are series of real numbers, so from calculus $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$. Hence,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0. \qquad \blacksquare$$

Corollary If $\sum_{n=1}^{\infty} z_n$ converges, then there exists $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$. That is, the sequence z_n is bounded.

Proof. If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$. Choose $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|z_n| < 1$. Then put

$$M = \max\{1, |z_1|, |z_2|, \dots, |z_{n_0-1}|\}.$$

Then $|z_n| \leq M$ for all $n \in \mathbb{N}$. ∎

Definition (Absolute Convergence) A series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ of real numbers converges.

Corollary (Absolutely Convergent Series Converge) If $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then it is convergent.

Proof. By assumption, the series $\sum_{n=1}^{\infty} |z_n|$ converges. Notice that $|x_n| \leq |z_n|$ and $|y_n| \leq |z_n|$ for all $n \in \mathbb{N}$. By the comparison test from calculus, the series

$$\sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n|$$

converge. Hence, the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are absolutely convergent and hence (by calculus) converge. By the proposition, we can conclude that $\sum_{n=1}^{\infty} z_n$ converges. ∎

Definition (Remainder of a Convergent Series) Suppose $\sum_{n=1}^{\infty} z_n$ is a convergent series and S its sum. The N^{th} remainder of the series is the complex number

$$\rho_N = S - S_N = S - \sum_{n=1}^{N} z_n = \sum_{n=1}^{\infty} z_n - \sum_{n=1}^{N} z_n.$$

//

The remainder provides a convenient way to prove that $\sum_{n=1}^{\infty} z_n = S$. Just notice that

$$|S_N - S| = |\rho_N - 0|$$

So $\sum_{n=1}^{\infty} z_n = S$ if and only if $\lim_{N\to\infty} \rho_N = 0$. We will frequently make use of this.

//

Power Series

Definition (Power Series) A power series is a series of

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where a_n is a sequence, $z_0 \in \mathbb{C}$ fixed, and z is any complex number in a prescribed region in $\mathbb{C}$. The associated sum, partial sum, and remainder depend on z, and are denoted

$$S(z), \quad S_N(z), \text{ and } \rho_N(z)$$

respectively.

Example (Geometric Series) We show that the geometric series $\sum_{n=0}^{\infty} a z^n$ is convergent when $|z|<1$. In fact,

$$\sum_{n=0}^{\infty} a z^n = \frac{a}{1-z} \qquad (|z|<1).$$

We compute the remainder:

$$\rho_N(z) = S(z) - S_N(z) = \frac{a}{1-z} - \sum_{n=0}^{N-1} a z^n$$

$$\left(1 + w + w^2 + \cdots w^{n-1} = \frac{1-w^n}{1-w}\right) \quad = \frac{a}{1-z} - a\left(\frac{1-z^N}{1-z}\right)$$

$$= a\left(\frac{z^N}{1-z}\right).$$

Hence, $|\rho_N(z)| = |a| \dfrac{|z|^N}{|1-z|}$. But the sequence of real numbers $\dfrac{|a|\,|z|^N}{|1-z|}$ converges to 0 if $|z|<1$ and diverges otherwise. Hence $\lim\limits_{N\to\infty} \rho_N(z) = \begin{cases} 0, & |z|<1 \\ \text{diverges otherwise.} \end{cases}$ //

$\boxed{\text{Theorem (Taylor's Theorem)}}$ Suppose that f is analytic on an open disk $D_R(z_0)$. Then at each $z \in D_R(z_0)$, $f(z)$ has a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

with coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

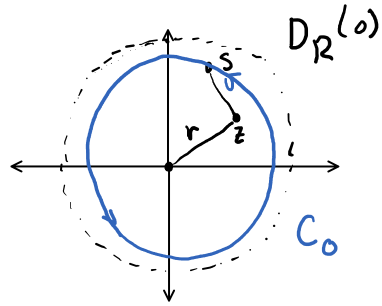
The series representation of f guaranteed by the theorem is called the Taylor Series of f about z_0.

Proof. First, assume that $z_0 = 0$ so that f is analytic $D_R(0)$.

Let $z \in D_R(0)$. Write $|z| = r$. Let r_0 be a real number such that $r < r_0 < R$. Let C_0 be the circle of radius r_0 centered at 0.

By Cauchy's Integral Formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z}\, ds.$$



Recall the formula:

$$1 + w + w^2 + \cdots w^{N-1} = \frac{1-w^N}{1-w} = \frac{1}{1-w} - \frac{w^N}{1-w}.$$

For any $N \in \mathbb{N}$, we can write

$$\frac{1}{s-z} = \frac{1}{s}\left(\frac{1}{1-\frac{z}{s}}\right) = \frac{1}{s}\left(\sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right)$$

$$= \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{s^N(s-z)}.$$

We compute the remainder:

$$\rho_N(z) = f(z) - \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z}\, ds - \sum_{n=0}^{N-1} \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{2\pi i} \int_{C_0} \frac{f(s)}{(s-0)^{n+1}} z^n\, ds$$ (Generalized Cauchy Integral Formula)

$$= \frac{1}{2\pi i}\int_{C_0} f(s)\left(\frac{1}{s-z} - \sum_{n=0}^{N-1} \frac{z^n}{(s-0)^{n+1}}\right) ds$$

$$= \frac{1}{2\pi i}\int_{C_0} f(s) \frac{z^N}{s^N(s-z)}\, ds$$ (by the formula for $\frac{1}{s-z}$).

Now, we prove that $\lim_{N\to\infty} \rho_N(z) = 0$. We have

$$|\rho_N(z)| = \frac{1}{2\pi}\left|\int_{C_0} f(s)\frac{z^N}{s^N(s-z)}\, ds\right|$$

$$\le \frac{1}{2\pi} \max_{s \in C_0}\left|\frac{f(s)\, z^N}{s^N(s-z)}\right| \cdot 2\pi r_0$$

$$= \max_{s \in C_0} \frac{|f(s)|\,|z|^N}{|s|^N\,|s-z|} \cdot r_0$$

$|s-z| \geq ||s|-|z||$
$= |r_0 - r|$
$= r_0 - r$

$$\leq \frac{r^N \cdot r_0}{r_0^N (r_0 - r)} \max_{s \in C_0} |f(s)|$$

Just a constant

The sequence $\left(\frac{r}{r_0}\right)^N \cdot \frac{r_0}{r_0 - r} M$ converges to 0 since $\frac{r}{r_0} < 1$.

This proves $\lim_{N \to \infty} \rho_N(z) = 0$. This proves the claim when $z_0 = 0$.

Now assume that $z_0 \neq 0$ so that f is analytic on the disk $D_R(z_0)$. Then $f(z + z_0)$ is analytic on $D_R(0)$. By the preceding case, we can write

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$$

Now replace z with $z - z_0$ to get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n .$$

This completes to proof. ∎

The Taylor series of f about $z_0 = 0$ is commonly referred to as a **MacLauren series**.

Examples (MacLauren Series of Elementary Functions) We will derive the following MacLauren Series representations of the most common elementary functions. We will frequently use these to compute MacLauren and Taylor series for other functions. You should memorize them!

(1) $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $(|z|<1)$

(2) $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$, $(|z|<\infty)$

(3) $\sin z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n+1}}{(2n+1)!}$, $(|z|<\infty)$

(4) $\cos z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n}}{(2n)!}$, $(|z|<\infty)$

(5) $\sinh z = \sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+1)!}$, $(|z|<\infty)$

(6) $\cosh z = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!}$, $(|z|<\infty)$

Solution.

(1) $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. Let $f(z) = \frac{1}{1-z}$. Then f has a singularity at $z=1$ so f is analytic on the disk $D_1(0)$. By Taylors theorem, f has a Maclaurin series on that disk. We have

$$f'(z) = \frac{d}{dz}(1-z)^{-1} = -1(1-z)^{-2}\cdot -1$$
$$= \frac{1}{(1-z)^2}$$

$$f''(z) = \frac{d}{dz}(1-z)^{-2} = -2\cdot(1-z)^{-3}\cdot -1$$
$$= \frac{2}{(1-z)^3}$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

Hence, $f^{(n)}(0) = n!$. Hence

$$\frac{1}{1-z} = f(z) = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n.$$

(2) $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$. Since $f(z) = e^z$ is entire, it has a

MacLauren series everywhere, by Taylors theorem. We have

$$f^{(n)}(0) = e^0 = 1.$$

Hence, $$e^z = f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(3) $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. We have

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{i^n z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-i)^n z^n}{n!}\right) \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{i^n z^n}{n!}\left(1 - (-1)^n\right) \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} i^{2n+1} \frac{z^{2n+1}}{(2n+1)!} \cdot 2 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.
\end{aligned}$$